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Transition to Effect Algebras

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An account is given of the recent development of the theory of effect algebras, their connection with partially ordered abelian groups, and their use for the mathematical representation of fuzzy or unsharp events. We submit an annotated list of important open problems, appropriate research projects, and unresolved philosophical issues engendered by the developing theory.

1. INTRODUCTION

At the September 1992 meeting of the International Quantum Structures Association in Castiglioncello, Italy, one of us (R.J.G.) presented a paper entitled, "The Transition to Orthoalgebras." Although the notion that orthoalgebras are true quantum logics had few adherents at that time, it has recently gained wide support. In the same spirit, we now commend effect algebras (or, what are the same things, D-posets) to the attention of quantum logicians.

As C. H. Randall taught us, it is a serious mistake [and perhaps even a "metaphysical disaster" (Randall and Foulis, 1983)] to regard all propositions pertaining to a physical system as forming a *single unified logical entity.* Indeed, propositions about a physical system naturally arrange themselves in a hierarchy of related, but distinct, logical structures. Only the propositions at the most basic level, those that are *experimentally testable, two-valued,* and *sharp,* but perhaps *unstable* (i.e., their truth values may change from trial to trial of an experiment) are represented by elements of an orthoalgebra (Foulis *et al.*, 1992). For instance, in the Fréchet–Kolmogorov[2] approach to

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[2] Kolmogorov himself carefully points out (Kolmogorov, 1933, p. v) that the representation of events by elements of a σ-algebra of sets is actually due to Fréchet (1930). It seems unfortunate that Kolmogorov alone is traditionally credited with this important development.

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probability theory, the sharp, yes/no, experimental propositions asserting that an event occurs are represented by elements of a σ-field of sets.

Von Neumann (1955, p. 253) observed that, "the relation between the *properties* of a physical system on the one hand, and the projections on the other, makes possible a sort of logical calculus with these" (our italics). In a slightly later paper Birkhoff and von Neumann (1936) speak about the *experimental propositions* associated with a physical system, with no further mention of properties of the system. This abrupt transition from properties to experimental propositions (which was never really explained by the authors) may be partially responsible for the unfortunate inclination of many quantum logicians to identify the lattice $\mathcal{L}$ of properties of a physical system $\mathcal{S}$ with the orthoalgebra L of testable, sharp, two-valued experimental propositions about $\mathcal{S}$.

In the first place, if $P \in \mathcal{L}$ is a possible property or attribute of a physical system $\mathcal{S}$, it may be far from clear when there is another property $P' \in \mathcal{L}$ that ought to be regarded as a *logical negation* of P. Being black is a property of ravens. Is being nonblack (as opposed to, say, being yellow or green) a *bona fide* property of canaries? Perhaps Hempel's (1965) paradox warns us against supposing that it is. Although the orthoalgebra L is orthocomplemented, there is no compelling reason to assume that the property lattice $\mathcal{L}$ carries a logically meaningful orthocomplementation. Mielnik (1974, 1976) seems to have been among the first to realize that the lattice of properties of a quantum mechanical system need not be orthocomplemented.

Second, although the propositions $p \in L$ are experimentally testable by definition, it may or may not be possible to design an experiment guaranteed in advance to ascertain with certainty whether or not a physical system has a particular property $P \in \mathcal{L}$. Piron (1981) gives a trenchant analysis of this critical distinction between properties on the one hand and two-valued experimental propositions (questions) on the other.

As well as the possible properties (or attributes) $P \in \mathcal{L}$ of the physical system $\mathcal{S}$ and the experimental propositions (or questions) $p \in L$ concerning $\mathcal{S}$, there are any number of additional kinds of statements or propositions that pertain to $\mathcal{S}$. These range from statistical or scientific hypotheses to assertions involving the relation between the mind of the observer and the system under observation. Lately, attention has begun to focus on fuzzy or unsharp propositions involving $\mathcal{S}$ (Mesiar, 1993) and this is where effect algebras come into the picture.

2. EFFECT ALGEBRAS

The prototype for the mathematical structures known as effect algebras is the set $\mathcal{E}(\mathcal{H})$ of all self-adjoint operators A on a Hilbert space $\mathcal{H}$ such that

$\mathbb{O} \leq A \leq \mathbb{1}$. In Ludwig's (1986) approach to the foundations of quantum mechanics, elements of $\mathscr{E}(\mathscr{H})$ are called *effects*.

If $A, B \in \mathscr{E}(\mathscr{H})$, we say that $A \oplus B$ is defined iff $A + B \leq \mathbb{1}$, in which case we define $A \oplus B := A + B$. (We use $:=$ to mean "equals by definition.") Thus, the *standard effect algebra* $\mathscr{E}(\mathscr{H})$ satisfies the conditions in the following definition.

2.1. Definition. An *effect algebra* is an algebraic system $(E, 0, u, \oplus)$ consisting of a set E, two special elements $0, u \in E$ called the *zero* and the *unit*, and a partially defined binary operation $\oplus$ on E that satisfies the following conditions for all $p, q, r \in E$:

- (i) [*Commutative law*] If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.
- (ii) [*Associative law*] If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ is defined, $(p \oplus q) \oplus r$ is defined, and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.
- (iii) [*Orthosupplementation law*] For every $p \in E$ there exists a unique $q \in E$ such that $p \oplus q$ is defined and $p \oplus q = u$.
- (iv) [*Zero–unit law*] If $u \oplus p$ is defined, then $p = 0$.

Effect algebras are mathematically equivalent to the weak orthoalgebras of Giuntini and Greuling (1989) and to the D-posets of Kôpka and Chovanec (1994). For simplicity, we often refer to E, rather than to $(E, 0, u, \oplus)$, as being an effect algebra.

2.2. Definition. Let E be an effect algebra with unit u and let $a, b, c \in E$.

- (i) We say that a and b are *orthogonal* and write $a \perp b$ iff $a \oplus b$ is defined.
- (ii) E is said to be *coherent* iff $a \perp b$, $a \perp c$, and $b \perp c$ implies that $(a \oplus b) \perp c$.
- (iii) If $a \neq 0$ and $a \perp a$, then a is called an *isotropic* element of E.
- (iv) The unique element $c \in E$ such that $a \perp c$ and $a \oplus c = u$ is called the *orthosupplement* of a and is written as $a' := c$.
- (v) We write $a \leq b$ iff there exists $c \in E$ with $a \perp c$ and $a \oplus c = b$.

It is shown in Foulis and Bennett (1994) that an effect algebra E is partially ordered by the relation $\leq$ in part (v) of Definition 2.2, that $0 \leq a \leq u$ for all $a \in E$, and that $a \mapsto a'$ is an order-reversing involution on E with $0' = u$ and $u' = 0$. Furthermore, an orthoalgebra is the same thing as an effect algebra with no isotropic elements, and an orthomodular poset is the same thing as a coherent effect algebra.

If E is an effect algebra and $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ is a finite sequence of (not necessarily distinct) elements of E, then, by recursion, we say that

$a_1 \oplus a_2 \oplus a_3 \oplus \cdots \oplus a_{n-1} \oplus a_n$ is defined in E iff $s := a_1 \oplus a_2 \oplus a_3 \oplus \cdots \oplus a_{n-1}$ is defined in E and $s \oplus a_n$ is defined in E, in which case $a_1 \oplus a_2 \oplus a_3 \oplus \cdots \oplus a_{n-1} \oplus a_n := s \oplus a_n$. If (a_j) is an infinite sequence of elements in E, we say that $\oplus_j a_j$ is defined iff $s_n := a_1 \oplus a_2 \oplus a_3 \oplus \cdots \oplus a_n$ is defined for every $n \geq 1$ and $\{s_n | n \geq 1\}$ has a least upper bound s in E, in which case, $\oplus_j a_j := s$.

If E and F are effect algebras with units u and v, respectively, then a mapping $\phi: E \to F$ is called a *morphism* iff $\phi(u) = v$ and, for $a, b \in E$ with $a \perp b$, we have $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. A *$\sigma$-morphism* $\phi: E \to F$ is a morphism such that, whenever (a_j) is an infinite sequence of elements in E such that $\oplus_j a_j$ is defined in E, then $\oplus_j \phi(a_j)$ is defined in F and $\phi(\oplus_j a_j) = \oplus_j \phi(a_j)$.

The *positive operator-valued* (POV) measures used in the stochastic or phase-space approach to quantum mechanics (Busch *et al.*, 1991; Schroeck, 1994; Schroeck and Foulis, 1990) are σ-morphisms $\phi: E \to \mathscr{E}(\mathscr{H})$ from the Boolean σ-algebra E of measurable subsets of a Borel space to the standard effect algebra $\mathscr{E}(\mathscr{H})$ on a Hilbert space $\mathscr{H}$.

3. PARTIALLY ORDERED ABELIAN GROUPS AND INTERVAL EFFECT ALGEBRAS

By a *partially ordered Abelian group*, we mean an additively written Abelian group G equipped with a partial order relation $\leq$ that is *translation invariant* in the sense that, for $a, b, c \in G$, $a \leq b \Rightarrow a + c \leq b + c$. For such a group G, the subset $G^+ := \{g \in G | 0 \leq g\}$, called the *positive cone*, satisfies $G^+ + G^+ \subseteq G^+$ and $G^+ \cap -G^+ = \{0\}$. Conversely, if G^+ is a subset of the Abelian group G and G^+ satisfies the last two conditions, then there is one and only one way to organize G into a partially ordered Abelian group in such a way that G^+ is its positive cone, namely, by defining $a \leq b$ for $a, b \in G$ iff $\exists c \in G^+$ with $a + c = b$ (Goodearl, 1986).

If G is a partially ordered Abelian group and $0 \neq u \in G^+$, the interval $G^+[0, u] := \{g \in G | 0 \leq g \leq u\}$ can be organized into an effect algebra with unit u by defining $a \oplus b$ for $a, b \in G^+[0, u]$ iff $a + b \leq u$, in which case $a \oplus b := a + b$ (Bennett and Foulis, n.d.). An effect algebra of the form $E = G^+[0, u]$ is called an *interval effect algebra* and G is called an *ambient group* for E. In general, the ambient group G is not uniquely determined by E.

3.1. Example. Let $\mathbb{Z}^+$ denote the standard positive cone in the additive group $\mathbb{Z}$ of integers and let $\mathbb{Z}_2$ be the additive group of integers modulo 2. Let $A := \mathbb{Z} \times \mathbb{Z}$ with positive cone $A^+ := \mathbb{Z}^+ \times \mathbb{Z}^+$, $B := \mathbb{Z} \times \mathbb{Z}_2$ with positive cone $B^+ := \{(n, \alpha) | 0 \neq n \in \mathbb{Z}^+\} \cup \{(0, 0)\}$, and $C := \mathbb{Z}$ with positive cone $C^+ := \{3n + 4m \mid n, m \in \mathbb{Z}^+\}$. Then the interval effect algebras $A^+[(0, 0), (1, 1)]$, $B^+[(0, 0), (1, 1)]$, and $C^+[0, 7]$ are mutually isomorphic.

If E is an effect algebra and K is an Abelian group, then a mapping $\phi: E \to K$ is called a *group-valued* measure iff, for $a, b \in E$ with $a \perp b$, $\phi(a \oplus b) = \phi(a) + \phi(b)$. It is shown in Bennett and Foulis (n.d.) that, if E is an interval effect algebra, there is a partially ordered Abelian group G and an element $0 \neq u \in G^+$ such that $E = G^+[0, u]$, $G = G^+ - G^+$, every element in G^+ is a sum of a finite sequence of elements in E, and every K-valued measure $\phi: E \to K$ can be extended uniquely to a group homomorphism $\phi^*: G \to K$. The group G, which is uniquely determined by E up to an isomorphism, is called the *universal group* of G.

3.2. Example. If $\mathcal{H}$ is a Hilbert space, then the additive Abelian group ϑ of all self-adjoint operators on $\mathcal{H}$ with positive cone ϑ^+ consisting of the positive-semidefinite operators in ϑ is the universal group for the standard effect algebra $\mathcal{E}(\mathcal{H}) = \vartheta^+[0, 1]$.

3.3. Example. The additive group $\mathbb{R}$ of real numbers with the standard positive cone $\mathbb{R}^+$ is the universal group for the interval effect algebra $\mathbb{R}^+[0, 1]$, called the *standard scale.*

A *probability measure* (or state) on an effect algebra E is the same thing as a morphism $\phi: E \to \mathbb{R}^+[0, 1]$ of E into the standard scale. Every interval effect algebra admits at least one probability measure, and an effect algebra with an order-determining set of probability measures is an interval effect algebra (Bennett and Foulis, n.d.). Thus, most of the orthostructures that have been seriously proposed as models for quantum logics are, in fact, interval effect algebras.

The basic theory of partially ordered Abelian groups was developed between 1930 and 1950 by Clifford, Birkhoff, Everett, Freudenthal, Fuchs, Kantorovitch, Levi, Riesz, Stone, Ulam, *et al.* Over the past two decades the subject has enjoyed a vigorous renaissance owing to the work of Effros, Elliott, Ellis, Goodearl, Handelman, Murphy, Shen, *et al.* Much of the renewed interest in partially ordered Abelian groups derives from their connections with operator algebras. For instance, if A is the additive group of a unital C^*-algebra with the positive cone $A^+ := \{aa^* \mid a \in A\}$, then $A^+[0, 1]$ is an interval effect algebra generalizing the standard effect algebra $\mathcal{E}(\mathcal{H})$.

Also, if R is a ring with unit and $G = K_0(R)$ is the Grothendieck group, the subset $G^+ \subseteq G$ consisting of all stable isomorphism classes in the category of finitely generated projective right R-modules is often a cone in G, and the stable isomorphism class $[R]$ is a natural order-unit in the resulting partially ordered Abelian group (Goodearl *et al.*, 1980). An indication of how $K_0(R)$ is exploited to investigate the structure of R is found in an expository paper of Murphy (1992). For a more detailed account, consult the monograph of Goodearl (1986).

4. RESEARCH PROJECTS AND OPEN PROBLEMS

The theory of effect algebras has been under development for only a short time and there is some uncertainty about the ends toward which future research efforts should be directed. This uncertainty is compounded by the richness and fecundity of the part of the mathematical theory already formulated and its many connections with various branches of mathematics and physics. Thus, in the best scientific tradition, it may be useful to provide a compilation of appropriate research projects, important open problems, and philosophical issues that need to be resolved. In this and the next section, we submit a short annotated list of projects and problems that we feel deserve some attention.

We begin with what, from the standpoint of quantum logic, has to be regarded as the *premier project.*

4.1. Project. Provide a basis for deciding *precisely when* and *exactly how* an experimental, observational, or operational situation, either real or idealized and either practical or contrived, gives rise to events, questions, propositions, or observables that can be regarded as fuzzy or unsharp and that are represented by elements of an effect algebra.

The classical notion of a sample space (Feller, 1950) and its generalization to a test space (Foulis *et al.*, 1993) or manual of experiments (Randall and Foulis, 1973) provide just such a basis for identifying sharp yes/no propositions affiliated with an experimental context and generate a representation for these propositions as elements of an orthoalgebra. The D-test spaces of Dvurečenskij and Pulmannová (1994) are mathematical structures related to effect algebras as test spaces are related to orthoalgebras; however, the problem of identifying elements of an appropriate D-test space with unsharp events in an experimental context has yet to be dealt with. For instance, it ought to be possible to associate an appropriate D-test space and a corresponding effect algebra with the quantum probability models of Aerts (1994).

The concepts of unsharpness or fuzziness that have hitherto influenced the study of effect algebras have arisen, on the one hand, in stochastic (or phase-space) quantum mechanics (Schroeck, 1994) and, on the other hand, in fuzzy set theory (Zadeh, 1965). Ideally, this project should accommodate (and perhaps unify) both these sources of motivation.

Much has been written about fuzzy sets as generalized "characteristic functions" taking on values in the unit interval $\mathbb{R}^+[0, 1]$ rather than in $\{0, 1\}$, and it is a simple matter to form effect algebras with such functions as their elements.

However, it is not so clear, either in principle or in practice, how to represent the obviously fuzzy events affiliated with specific experimental situations (e.g., a nearsighted person reporting the outcomes of dice rolls at a craps table) as elements of an effect algebra.

H. Lamb, in a 1904 British Association presidential address, commented on experimental fuzziness as follows (Lamb, 1904): "The more refined the methods employed the more vague and elusive does the supposed magnitude become; the judgment flickers and wavers, until at last in a sort of despair *some* result is put down, not in the belief that it is exact, but with the feeling that it is the best we can make of the matter." Modern digital readouts only sweep Lamb's concerns under a carpet of electronic instrumentation. Ideally, the proposed project should clarify the relations among *all* sources of experimental uncertainty, unsharpness, and fuzziness, including Heisenberg uncertainty, unsharpness in stochastic quantum mechanics, and the observational vagaries to which Lamb alludes.

In connection with this project, it may be useful to note that, by forming a tensor product $E := S \otimes A$ of a scale algebra S and an orthoalgebra A, one obtains an effect algebra E that is generally not an orthoalgebra (Foulis *et al.*, 1994). This is one of the ways (and perhaps, from the quantum logic point of view, the only reasonable way) to impose "fuzziness" on the otherwise "sharp" propositions in A.

4.2. Project. Form a suitable combination of the axioms for a BZ-poset (Cattaneo and Nistico, 1989) and the axioms for an effect algebra to produce an appropriate notion of a *BZ-effect algebra.*

In conducting this project, the standard effect algebra $\mathscr{E}(\mathscr{H})$ for a Hilbert space could serve as a paradigm. More generally, the unit interval $\mathscr{A}^+[0, 1]$ in a unital von Neumann algebra $\mathscr{A}$ should form a BZ-effect algebra in a natural way and the salient features of $\mathscr{A}^+[0, 1]$ should be reflected in general theorems for BZ-effect algebras. The notion of an *effect ring*, introduced in Greechie *et al.* (n.d.), could provide a starting point for this project. Also, it might be desirable to introduce additional axioms suggested by the convexity structure of $\mathscr{A}^+[0, 1]$.

4.3. Project. Create an appropriate definition and a corresponding theory of the "sharp" effects in a general effect algebra.

If $\mathscr{H}$ is the Hilbert space for a quantum mechanical system $\mathscr{S}$, the (possibly) unsharp experimental propositions concerning $\mathscr{S}$ are represented by effect operators on $\mathscr{H}$, whereas the sharp propositions correspond to projection operators on $\mathscr{H}$. The projection operators form a sub-effect algebra $\mathbb{P}(\mathscr{H})$ of the standard effect algebra $\mathscr{E}(\mathscr{H})$, and there are various ways of characterizing elements of $\mathbb{P}(\mathscr{H})$ among the elements of $\mathscr{E}(\mathscr{H})$. For instance, if $A \in \mathscr{E}(\mathscr{H})$, then $A \in \mathbb{P}(\mathscr{H})$ iff $A \wedge A' = \mathbb{O}$ (Dvurečenskij, n.d.; Gudder, n.d.; Greechie *et al.*, n.d.). Given a general effect algebra E, how can we single out a sub-effect algebra P consisting of the "sharp" elements of E? Presumably, P should be an orthoalgebra containing the center of E (Greechie *et al.*, n.d.) and every element $a \in P$ should satisfy (at least) the condition $a \wedge a' = 0$.

In this regard, the following is of interest: In the theory of partially ordered Abelian groups G with order unit u, an element $a \in G^+[0, u]$ is said to be *characteristic* iff $a \wedge (u - a) = 0$ in $G^+[0, u]$. If G is lattice ordered, or even an interpolation group, then the characteristic elements form a Boolean sub-effect algebra of the interval effect algebra $G^+[0, u]$ (Goodearl, 1986; Goodearl *et al.*, 1980).

4.4. Project. Study and classify ideals in effect algebras and investigate quotients of effect algebras modulo suitable ideals.

The quotient of an orthomodular lattice modulo a p-ideal (Kalmbach, 1983) serves as a prototype for the general definition. For interval effect algebras, the resulting theory ought to relate to the existing theory of ideals and quotients in partially ordered Abelian groups (Goodearl, 1986, pp. 8–11).

The motivating idea is as follows: If E is the effect algebra of (possibly) unsharp propositions affiliated with a physical system $\mathcal{S}$, and if it is known that $\mathcal{S}$ is in a certain physical state ψ, the propositions $e \in E$ that are *impossible* in state ψ should form an ideal I in E. The remaining propositions in $E \backslash I$, those that are still *possible* in state ψ, should generate the quotient effect algebra E/I in an appropriate way.

For instance, suppose $\mathcal{S}$ is a pair of dice at a craps table and E is the Boolean algebra formed by the 2^{11} (sharp) events for the sample space $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ of outcomes for a roll of the dice. If the dice are loaded so that only the outcomes 7 and 11 can obtain, then the events contained in the set $\{2, 3, 4, 5, 6, 8, 9, 10, 12\}$ form the ideal I of impossible elements of E, and the quotient E/I is canonically isomorphic to the Boolean algebra formed by the 2^2 events for the sample space $\{7, 11\}$.

4.5. Project. Study how properties of an interval effect algebra E relate to properties of its ambient groups, and, in particular, to its universal group G.

For instance, what properties of E correspond to the condition that G be lattice ordered, Archimedean, unperforated, torsion free, or has the interpolation property? For example, if G is lattice ordered, it is clear that E is lattice ordered as well; however, the universal group of the orthomodular lattice G_{12} (Beltrametti and Cassinelli, 1981, p. 101) is not even an interpolation group.

The class of χ-algebras, introduced in Foulis *et al.* (1994), deserves further study as part of this project. A χ-algebra is an interval effect algebra whose universal group is $G = \mathbb{Z}^r$ for some finite positive integer r, where G^+ is a subcone of the standard positive cone $(\mathbb{Z}^+)^r$, and the order-unit is $u := (1, 1, 1, \ldots, 1)$. For instance, G_{12} is a χ-algebra. The class of χ-algebras is closed under Cartesian products, horizontal sums, and tensor products.

Another important part of this project derives from the case in which R is a unital ring, G is the algebraic K-group $K_0(R)$, and the stable isomorphism class $[R]$ is taken as the order-unit for the interval effect algebra E (Goodearl *et al.*, 1980; Murphy, 1992). Here we have a host of challenging problems; for instance, what properties of E correspond to the condition that R be commutative, Noetherian, unit regular, an AFC^*-algebra, or an AW^*-algebra?

4.6. Project. Establish connections with the base-norm order-unit approach to the foundations of quantum mechanics.

Many of the connections between base-norm and their dual order-unit spaces and partially ordered Abelian groups with order-unit are already known (Goodearl, 1986), so this project is closely related to Project 4.5. As a starting point, we have the following result (Foulis and Bennett, 1994): Let E be an interval effect algebra with universal group G and let $V(E)$ be the real vector space of all $\mathbb{R}$-valued measures (i.e., charges) on E. Then, if $V(E)$ is finite dimensional, so is the real vector space $\mathbb{R} \otimes G$, and $\mathbb{R} \otimes G$ is canonically isomorphic to the dual space of $V(E)$.

4.7. Project. Clarify the relationship between effect algebras and orthosupplemented posets.

By an orthosupplemented poset, we mean a bounded poset E equipped with an order-reversing involution $e \mapsto e'$. In the terminology of Gudder, an orthosupplemented poset is a semiorthoposet in which every element is closed (Gudder, 1994). Which orthosupplemented posets can be organized into effect algebras? When can a poset with involution be organized into an effect algebra in more than one way? Characterize those effect algebras E that, as a poset, have various properties of interest, e.g., $(E, \leq)$ is a distributive lattice, a modular lattice, a Heyting lattice, or a semimodular lattice. A starting point is the theorem of Pulmannová, Greechie, and Foulis characterizing the finite effect algebras that form a distributive lattice (Greechie *et al.*, n.d.).

5. LIFTING PROBLEMS FOR MORPHISMS OR MEASURES

By a *lifting problem* is meant a problem as illustrated in Fig. 1. Here A, B, C are mathematical structures, ϕ and ι are mappings, and the problem is to find a mapping ϕ^* that makes the diagram commutative in the sense

$$\begin{tikzcd} C \arrow[dr, "\phi^*"] & \\ A \arrow[u, "\iota"] \arrow[r, "\phi"] & B \end{tikzcd}$$

Fig. 1.

that $\phi = \phi^* \circ \iota$. If ϕ^* can be found, it is said to be obtained by *lifting* ϕ *through* ι to C. If $A \subseteq C$ and $\iota(a) = a$ for all $a \in A$, we say that ϕ^* is an extension of ϕ. (If ι is an injection, we may wish to regard ϕ^* as an extension of ϕ in disguise.)

Here we are suggesting the very general project of *systematically studying lifting and extension problems that involve effect algebras.* To begin with, it seems desirable to consider some of the known lifting and extension theorems involving special types of effect algebras. These theorems need to be classified, related, and possibly extended to more general effect algebras. In what follows, we attempt to launch this project by submitting a list of some of the more important lifting and extension theorems.

There are two variations of the basic lifting problem. We begin with the *first variation* in which A, B, C, ϕ, and ι are given, and the only problem is to find ϕ^*, if it exists.

5.1. Example (Extension of Measures). Let C be a field of subsets of a set X, let A be a subfield of C, let $\iota: A \to C$ be the inclusion mapping, suppose that B is an Abelian group, and let $\phi: A \to B$ be a B-valued measure on A. The problem of finding $\phi^*: C \to B$ is the problem of extending the measure ϕ on A to a measure ϕ^* on C.

Carlson and Prikry (1982) have shown that the problem in Example 5.1 has a positive solution when C is the power set of X and A is any field of subsets of X. Related results can be found in Bhaskara Rao and Shortt (1992).

5.2. Example (Integration). Let A be a field of subsets of a set X, let $B = \mathbb{R}$, let $\phi: A \to B$ be a measure on A, let C be a subalgebra of the algebra of all measurable functions on X, suppose that C contains all simple functions on X, and let $\iota(E) = \chi_E$, the characteristic set function of E, for all $E \in A$. Then the problem of finding ϕ^* is precisely the problem of finding the integral $\phi^*(f) = \int_X f(x)\, d\phi(x)$, when it exists.

Of course, there are many special cases (e.g., A is a σ-field, ϕ is σ-additive, etc.) and alternative versions (e.g., $B = \mathbb{R} \cup \{\infty\}$, ϕ is an extended real-valued function, etc.) of Example 5.2.

5.3. Example (Trace). Let C be an irreducible von Neumann algebra of finite type, let A be the complete orthomodular lattice of projections in C, let $\iota: A \to C$ be the inclusion mapping, let $B = \mathbb{R}$, and let $\phi: A \to \mathbb{R}^+[0,,1] \subseteq B$ be the normalized dimension function on A. Then $\phi^*: C \to \mathbb{R}$ is the trace function on C.

Again, there are many generalizations and related versions of Example 5.3. For instance, if C is reducible, B can be replaced by the algebra of all continuous real-valued functions on the Stone space of the center of A. A closely related result is the following (Bunce and Wright, 1992; Dvurečenskij, 1993).

5.4. Example (Mackey, Gleason, Bunce, Wright Theorem). Let C be a von Neumann algebra with no type I_2 direct summand, let A be the complete orthomodular lattice of projections in C, let $\iota: A \to C$ be the inclusion mapping, let B be a Banach space, and suppose that $\phi: A \to B$ is a bounded vector-valued measure on A. Then there is a bounded linear operator $\phi^*: C \to B$ such that $\phi = \phi^* \circ \iota$.

For the *second variation* of lifting/extension questions, only A, B, and ϕ are given, and the problem is to find C, ι, and ϕ^*. In the second variation, it is usually understood that C is to be more amenable to mathematical analysis than A. Here are some examples.

5.5. Example (Carathéoradory Extension). This is a variation of Example 5.1 in which $B = \mathbb{R} \cup \{\infty\}$, $\phi: A \to B$ is σ-subadditive, and C is not given *a priori*, but has to be constructed as a σ-field on which ϕ^* is a σ-additive measure.

An account of the Carathéoradory extension theorem can be found in Hewett and Stromberg (1965, pp. 126–127).

5.6. Example (The Universal Group). If A is an effect algebra, there is a group C and a C-valued measure $\iota: A \to C$ such that $\iota(A)$ generates C and, for every group B and every B-valued measure $\phi: A \to B$, there is a group homomorphism $\phi^*: C \to B$ such that $\phi = \phi^* \circ \iota$.

The construction of the universal group can be found in Bennett and Foulis (n.d.) and Foulis and Bennett (1994).

5.7. Example (Naimark Extension). Let A be a σ-field of subsets of X, let B be the standard effect algebra $\mathscr{E}(\mathscr{H})$ on the Hilbert space $\mathscr{H}$, and let

$\phi: A \to B$ be a σ-complete effect-algebra morphism (i.e., a POV measure). By the Naimark extension theorem, it is possible to find a Hilbert space $\mathcal{K}$ which is an extension of $\mathcal{H}$ and a projection-valued measure $\iota: A \to C := \mathbb{P}(\mathcal{K})$ such that $\phi = \phi^* \circ \iota$, where $\iota: C \to B$ is the natural restriction mapping.

A proof of the Naimark Extension Theorem can be found in Naimark (1943) and Riesz and Sz.-Nagy (1960).

5.8. Example (Wigner–Wright Theorem). Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces over $\mathbb{R}$, let $A := \mathbb{P}(\mathcal{H})$ and $B := \mathbb{P}(\mathcal{K})$ be the complete orthomodular lattices of projection operators on $\mathcal{H}$ and $\mathcal{K}$, respectively, and suppose that $\phi: A \to B$ is a σ-complete effect-algebra morphism. Then there is a Hilbert space $\mathcal{J}$ and a unitary isomorphism $U: \mathcal{H} \otimes \mathcal{J} \to \mathcal{K}$ such that $\phi = \phi^* \circ \iota$, where $\iota: A \to C := \mathbb{P}(\mathcal{H} \otimes \mathcal{J})$ is given by $\iota(P) := P \otimes 1_{\mathcal{J}}$ for all $P \in \mathbb{P}(\mathcal{H})$ and $\phi^*(Q) := UQU^{-1}$ for all $Q \in \mathbb{P}(\mathcal{H} \otimes \mathcal{J})$.

An account of the Wigner–Wright theorem, its generalization to complex Hilbert spaces, and its connection with Wigner's theorem on symmetry transformations can be found in Wright (1977). Although the Wigner–Wright theorem has received little attention from quantum logicians, it would seem to have important consequences. Essentially, it says that, if a first quantum mechanical system $\mathcal{S}_1$ with Hilbert space $\mathcal{H}$ can be "explained" by a second quantum mechanical system $\mathcal{S}_2$ with Hilbert space $\mathcal{K}$ (in the same sense that a thermodynamic system can be explained via statistical mechanics by a mechanical system), then $\mathcal{S}_2$ is just $\mathcal{S}_1$ coupled with a third quantum mechanical system $\mathcal{S}_3$ with Hilbert space $\mathcal{J}$. In other words, with orthodox quantum mechanical systems based strictly on Hilbert spaces, we have reached an explanative impasse!

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